On the Multiplicative Inverse Eigenvalue Problem*

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ABSTRACT

Let $A=(a_{ij})$ be an n-square matrix over an arbitrary field K, and let $w_1,\dots,w_n$ be elements of K. The following problem is well known: Under what conditions does there exist an n-square diagonal matrix D, over K, such that DA has eigenvalues $w_1,\dots,w_n$? Some solutions to the problem are known when K is the complex field $\mathbb{C}$. It was proved by Friedland that if the principal minors of A are not zero, then the matrix D exists. The result that we present is the extension to an arbitrary algebraically closed field of the sufficient conditions established by Friedland.

1. INTRODUCTION

Let $A=(a_{ij})$ be an n-square matrix over an arbitrary field K, and let $w_1,\dots,w_n$ be elements of K. The following problem is well known.

PROBLEM. Under what conditions does there exist an n-square diagonal matrix D, over K, such that

$$DA$$

has eigenvalues $w_1,\dots,w_n$?

Some solutions to this problem are known when K is the complex field $\mathbb{C}$. In [2,3] it was proved that if all the principal minors of A are not zero, then the matrix D exists. The result that we present in the sequel is the extension

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to an arbitrary algebraically closed field of the sufficient conditions established in [2,3]. Moreover our proof is purely algebraic, and its technique was inspired by [1].

2. PRELIMINARIES

Let $x_1,\ldots,x_n$, y be indeterminates and

$$D=\operatorname{diag}(x_1,\ldots,x_n).$$

Then (see, for instance, [8, pp. 21,22])

$$\det(yI-DA)=y^n+\sum_{t=1}^{n}(-1)^n y^{n-t} \times \sum_{1\leqslant i_1<\cdots<i_t\leqslant n}\det A[i_1,\ldots,i_t|i_1,\ldots,i_t]x_{i_1}\cdots x_{i_t}, \tag{1}$$

where

$$A[i_1,\ldots,i_t|i_1,\ldots,i_t]$$

is the principal submatrix of A contained in the rows $i_1,\ldots,i_t$.

The equality (1) shows that the conditions for the existence of a solution to our problem are the same as the conditions for the existence of a solution of the system of equations

$$\sum_{1\leqslant i_1<\cdots<i_t\leqslant n}\det A[i_1,\ldots,i_t|i_1,\ldots,i_t]x_{i_1}\cdots x_{i_t}=S_t(w_1,\ldots,w_n), \qquad t=1,\ldots,n, \tag{2}$$

where $S_1,\ldots,S_n$ are the elementary symmetric polynomials.

Following [1], let K be an algebraically closed field, and let $K[x_1,\ldots,x_n]$ be the ring of the polynomials over K in the indeterminates $x_1,\ldots,x_n$. Let $q>0$ and

$$R_i(x_1,\ldots,x_n)=x_i^q+P_i(x_1,\ldots,x_n), \qquad i=1,\ldots,n, \tag{3}$$

where

$$\deg P_i(x_1,\dots,x_n) < q, \qquad i=1,\dots,n.$$

We denote by $(R_1,\dots,R_n)$ the ideal generated by $R_1,\dots,R_n$. We make the convention that $\deg 0 = -\infty$ and $-\infty < n$ for any integer n.

THEOREM 1. *Any polynomial $Q(x_1,\dots,x_n)$ of $K[x_1,\dots,x_n]$ can be written as a polynomial expression*

$$Q^*(x_1,\dots,x_n,R_1,\dots,R_n) \tag{4}$$

of $x_1,\dots,x_n$, $R_1,\dots,R_n$, i.e., as a sum of monomials of the form

$$ax_1^{j_1}\cdots x_n^{j_n}R_1^{s_1}\cdots R_n^{s_n}, \qquad 0\leqslant j_1<q,\dots, \qquad 0\leqslant j_n<q.$$

Moreover the polynomial Q^* is unique.

THEOREM 2. *Let $P(x_1,\dots,x_n)$ be an nonzero polynomial of the form*

$$P(x_1,\dots,x_n)=\sum a_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}, \tag{6}$$

$0\leqslant i_1<q,\dots,0\leqslant i_n<q$. *Then $P(x_1,\dots,x_n)$ does not belong to the ideal $(R_1,\dots,R_n)$.*

These results were proved in [1].

THEOREM 3. *The system of polynomial equations*

$$x_i^q + P_i(x_1,\dots,x_n)=0, \qquad i=1,\dots,n, \tag{7}$$

where the degree of each $P_i(x_1,\dots,x_n)$ is smaller than q, has a solution over the algebraically closed field K. The number of solutions of (7) *is finite.*

Proof. The proof of this result is also in [1]. ■

3. PRINCIPAL RESULTS

LEMMA 1. *Let $f_t \in K[x_1,\ldots,x_n]$ (K algebraically closed), where*

$$f_t = \sum_{1 \leqslant i_1 < \cdots < i_t \leqslant n} a_{i_1 \cdots i_t} x_{i_1} \cdots x_{i_t}, \qquad t = 1,\ldots,n, \tag{8}$$

with $a_{i_1 \cdots i_t} \neq 0$, $t = 1,\ldots,n$, $1 \leqslant i_1 < \cdots < i_t \leqslant n$. There exist homogeneous polynomials h_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n$, and an integer q such that

$$x_i^q = h_{i1} f_1 + \cdots + h_{in} f_n, \qquad i = 1,\ldots,n, \tag{9}$$

$$\deg h_{ij} = q - j \text{ or } \deg h_{ij} = -\infty, \qquad i = 1,\ldots,n, \quad j = 1,\ldots,n, \tag{10}$$

$$x_i^p \mid h_{i,n-p}, \qquad i = 1,\ldots,n, \quad p = 1,\ldots,n-1. \tag{11}$$

Proof. Let E be an extension of K. We show that the polynomials $f_1,\ldots,f_n$ have a unique common zero over E the trivial zero $(0,\ldots,0)$. The proof is by induction. If $n = 1$ the result is clearly true.

We denote by $(f_t)_0$ the polynomial

$$\begin{aligned}(f_t)_0 &= f_t(x_1,\ldots,x_{n-1},0)\\ &= \sum_{1 \leqslant i_1 < \cdots < i_t \leqslant n-1} a_{i_1 \cdots i_t} x_{i_1} \cdots x_{i_t}, \qquad t = 1,\ldots,n-1.\end{aligned}$$

If $(e_1,\ldots,e_n)$ is a common zero of $f_1,\ldots,f_n$, then obviously one of the e_i's $(i = 1,\ldots,n)$ must be zero. We may assume, without loss of generality, that $e_n = 0$. By the induction hypothesis $(f_1)_0,\ldots,(f_{n-1})_0$ have as unique common zero the trivial zero. Therefore $(e_1,\ldots,e_{n-1})$ is a common zero of $(f_1)_0,\ldots,(f_{n-1})_0$. Thus

$$e_1 = \cdots = e_{n-1} = e_n = 0.$$

By a consequence of the Hilbert *Nullstellensatz* [15, p. 158] there exists an integer q' and homogeneous polynomials m_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n$, such that

(a) $x_i^{q'} = m_{i1} f_1 + \cdots + m_{in} f_n$, $i = 1,\ldots,n$,
(b) $\deg m_{ij} = q' - j$ or $\deg m_{ij} = -\infty$, $i = 1,\ldots,n$, $j = 1,\ldots,n$.

Let $q = q' + (n-1)$ and $h_{ij} = x_i^{n-1} m_{ij}$, $i = 1, \dots, n$, $j = 1, \dots, n$. It is clear that polynomials h_{ij} and the integer q satisfy the conditions (9), (10), (11). ■

REMARK. This proof goes further than the condition (11) states. We have proved that $x_i^{n-1} \mid h_{ij}$, $j = 1, \dots, n$, $i = 1, \dots, n$.

LEMMA 2. *If* $F(x_1, \dots, x_n) \in (x_1^q, \dots, x_n^q)$ *is homogeneous of degree* d $(0 \neq F(x_1, \dots, x_n) \in K[x_1, \dots, x_n])$, *then there exist polynomials* G *and* T *belonging to* $K[x_1, \dots, x_n]$ *such that*

$$F(x_1, \dots, x_n) = G(x_1, \dots, x_n) + T(x_1, \dots, x_n),$$

$$G(x_1, \dots, x_n) \in (R_1, \dots, R_n), \quad (12)$$

and

$$T(x_1, \dots, x_n) = \sum b_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}, \quad (13)$$

where $0 \leqslant k_1 < q, \dots, 0 \leqslant k_n < q$, *and* $\deg T < d$.

Proof. Let us consider a monomial $x_1^{i_1} \cdots x_n^{i_n}$, $i_1 + \cdots + i_n = d$, appearing in $F(x_1, \dots, x_n)$. By Theorem 1

$$x_1^{i_1} \cdots x_n^{i_n} = \sum a_{j_1 \cdots j_n s_1 \cdots s_n} x_1^{j_1} \cdots x_n^{j_n} R_1^{s_1} \cdots R_n^{s_n}$$

with $0 \leqslant j_1 < q, \dots, 0 \leqslant j_n < q$.

Since $F(x_1, \dots, x_n) \in (x_1^q, \dots, x_n^q)$, all the monomials $x_1^{i_1} \cdots x_n^{i_n}$ belong to $(x_1^q, \dots, x_n^q)$. Therefore one of the $i_1, \dots, i_n$ must be greater or equal to q, and so at least one of the R_i's does appear on the right hand side of the above equality. Let

$$\tilde{G}(x_1, \dots, x_n, R_1, \dots, R_n)$$

be the sum of the monomials containing at least one of the R_i's, and let $T(x_1, \dots, x_n)$ be the remaining part,

$$T(x_1, \dots, x_n) = \sum c_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

$$0 \leqslant k_1 < q, \dots, 0 \leqslant k_n < q. \quad (14)$$

We have

$$x_1^{i_1}\cdots x_n^{i_n}=\tilde{G}(x_1,\dots,x_n,R_1,\dots,R_n)+T(x_1,\dots,x_n). \qquad (15)$$

In $\tilde{G}+T$ pick a term of the form $bx_1^{j_1}\cdots x_n^{j_n}R_1^{s_1}\cdots R_n^{s_n}$ $(b\neq 0)$ (where the s_i's may be all zero, in which case the term is in T) for which the sum

$$(j_1+s_1q)+\cdots+(j_n+s_nq)$$

is maximal. Since this sum is maximal, the term $x_1^{j_1+s_1q}\cdots x_n^{j_n+s_nq}$ can come only from an appropriate term

$$x_1^{j_1}\cdots x_n^{j_n}R_1^{s_1}\cdots R_n^{s_n}, \qquad 0\leqslant j_1<q,\dots,\quad 0\leqslant j_n<q, \qquad (16)$$

and so it cannot be canceled out with any other term when we use (3). Therefore $x_1^{j_1+s_1q}\cdots x_n^{j_n+s_nq}$ is precisely $x_1^{i_1}\cdots x_n^{i_n}$. Thus in (16) it cannot be that $s_1=\cdots=s_n=0$, because then in $x_1^{i_1}\cdots x_n^{i_n}$ no exponent would be $\geqslant q$. On the right hand side of (15), after using (3), there cannot appear any other term of degree d, because it would not cancel out with any other and would be of the form $x_1^{h_1}\cdots x_n^{h_n}$ of degree d but with $(h_1,\dots,h_n)\neq(i_1,\dots,i_n)$. Such a term does not appear on the left hand side of (15), and so the degree of T is smaller than d.

Now using (3), (15) yields

$$x_1^{i_1}\cdots x_n^{i_n}=G(x_1,\dots,x_n)+T(x_1,\dots,x_n),$$

with $G\in(R_1,\dots,R_n)$ and T of the form (14), and the degree of T less than d.

The result for $F(x_1,\dots,x_n)$ is now obtained by summing the polynomials G and T of the monomials occurring in $F(x_1,\dots,x_n)$. ∎

LEMMA 3. *If* h_{ij}, $i=1,\dots,n$, $j=1,\dots,n$, *and* q *are in the conditions of Lemma* 1 *and*

$$R_i(x_1,\dots,x_n)=x_i^q+P_i(x_1,\dots,x_n), \qquad i=1,\dots,n,$$

with $\deg P_i(x_1,\dots,x_n)<q$, *then*

$$\det(h_{ij})\notin(R_1,\dots,R_n).$$

Proof. Let

$$\det(h_{ij}) = F_1(x_1,\dots,x_n) + F_2(x_1,\dots,x_n),$$

where F_1 is the sum of the monomials belonging to $(x_1^q,\dots,x_n^q)$ and F_2 is the sum of the monomials not belonging to $(x_1^q,\dots,x_n^q)$. Then F_2 will not belong to $(x_1^q,\dots,x_n^q)$ and will be of the form

$$F_2(x_1,\dots,x_n) = \sum a_{j_1\cdots j_n} x_1^{j_1}\cdots x_n^{j_n}, \qquad 0 \leqslant j_1 < q,\dots, \quad 0 \leqslant j_n < q.$$

We show by induction that $F_2 \neq 0$. For $n = 1$ the result is obvious. Let us assume that it is valid for $n-1$. Let

$$H = (h_{ij}).$$

We have

$$\begin{bmatrix} x_1^q \\ \vdots \\ x_n^q \end{bmatrix} = H \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}. \tag{17}$$

Let us set $x_n = 0$. Let $(f_t)_0 = f(x_1,\dots,x_{n-1},0)$, $\tilde{h}_{ij} = h_{ij}(x_1,\dots,x_{n-1},0)/x_i$ $(i, j = 1,\dots,n-1)$ [notice that x_i divides $h_{ij}(x_1,\dots,x_{n-1},0)$], and denote by $\tilde{H}$ the $(n-1)\times(n-1)$ matrix whose (i, j) element is $\tilde{h}_{ij}$. We obtain from (17)

$$\begin{bmatrix} x_1^{q-1} \\ \vdots \\ x_{n-1}^{q-1} \end{bmatrix} = \tilde{H} \begin{bmatrix} (f_1)_0 \\ \vdots \\ (f_{n-1})_0 \end{bmatrix}. \tag{18}$$

Observe that the polynomials $\tilde{h}_{ij}$ are homogeneous, $\deg \tilde{h}_{ij} = (q-1) - j$ or $\deg \tilde{h}_{ij} = -\infty$, and $x_i^{n-2} \mid \tilde{h}_{ij}$, $i = 1,\dots,n-1$, $j = 1,\dots,n-1$.

Let $\hat{h}_{ij} = h_{ij}/x_i$ $(i, j = 1,\dots,n)$, and denote by $\hat{H}$ the $n \times n$ matrix whose (i, j) element is $\hat{h}_{ij}$. Let $\hat{H}(i|j)$ be the matrix obtained from $\hat{H}$ by suppressing row i and column j. Notice that for $k \neq i$, x_k divides row k of $\hat{H}(i|j)$. Multiplying both sides of (17) by the adjoint of H on the left, and dividing by $x_1 \cdots x_n$, we obtain easily

$$a_{1\dots n}\det H = x_1^{q-1}(-1)^{n+1}\det \hat{H}(1|n) + \cdots + x_1^{q-1}\det \hat{H}(n|n). \tag{19}$$

Denote by $\hat{H}(n|n)_0$ the matrix obtained from $\hat{H}(n|n)$ by setting $x_n = 0$. Obviously $\hat{H}(n|n)_0 = \tilde{H}$. Since $\det \hat{H}(n|n)_0 = \det \tilde{H}$, the equality (18) gives by the induction hypothesis

$$\det \hat{H}(n|n)_0 = \tilde{F}_1(x_1,\dots,x_{n-1}) + \tilde{F}_2(x_1,\dots,x_{n-1})$$

with

$$\tilde{F}_1(x_1,\dots,x_{n-1}) \in \left(x_1^{q-1},\dots,x_{n-1}^{q-1}\right)$$

and

$$0 \neq \tilde{F}_2(x_1,\dots,x_{n-1}) = \sum c_{r_1\cdots r_{n-1}} x_1^{r_1}\cdots x_{n-1}^{r_{n-1}},$$

$0 \leqslant r_1 < q-1,\dots,\ 0 \leqslant r_{n-1} < q-1$.

It is clear that $\det \hat{H}(n|n)_0$ consists exactly of those monomials which already appeared in $\det \hat{H}(n|n)$ and did not contain x_n. This means that all the monomials appearing in $\det \hat{H}(n|n)_0$ [in particular $(0 \neq)\ c_{r_1\cdots r_{n-1}} x_1^{r_1}\cdots x_{n-1}^{r_{n-1}}$ for a certain $(n-1)$-tuple $(r_1,\dots,r_{n-1})$ satisfying $0 \leqslant r_i < q-1$] do appear in $\det \hat{H}(n|n)$. It follows that

$$c_{r_1\cdots r_{n-1}} x_n^{q-1} x_1^{r_1}\cdots x_{n-1}^{r_{n-1}} \tag{20}$$

occurs in $x_n^{q-1}\det \hat{H}(n|n)$. Since $r_1,\dots,r_{n-1} < q-1$, the monomial (20) occurs in $\det H$, because the monomials in the remaining summands of (19) are of the form

$$x_1^{j_1}\cdots x_n^{j_n}$$

with at least one of the j_i's, $1 \leqslant i \leqslant n-1$, greater than or equal to $q-1$. The occurrence of the monomial (17) in $\det H$ implies that $\det H \neq 0$ and $F_2 \neq 0$.

By (10), $\det H = \det(h_{ij})$ is a homogeneous polynomial of degree

$$d = \sum_{j=1}^{n} (q-j).$$

Consequently F_1 and F_2 are homogeneous polynomials of degree d. Lemma 2 guarantees the existence of polynomials G and T such that

$$F_1(x_1,\dots,x_n) = G(x_1,\dots,x_n) + T(x_1,\dots,x_n)$$

with $G \in (R_1,\dots,R_n)$, $\deg T < d$, and

$$T(x_1,\dots,x_n) = \sum b_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

with $0 \leqslant k_1 < q,\dots,\ 0 \leqslant k_n < q$. Observe that, by Theorem 2, if

$$F_2 + T \neq 0$$

then

$$F_2 + T \notin (R_1,\dots,R_n).$$

Since $F_2 \neq 0$ and $\deg F_2 = d$ and $\deg T < d$, we conclude that $F_2 + T \neq 0$. As a consequence

$$\det(h_{ij}) = \det H = G(x_1,\dots,x_n) + T + F_2$$

does not belong to $(R_1,\dots,R_n)$ [we recall that $G \in (R_1,\dots,R_n)$]. ■

Before stating our next theorem we define "almost everywhere." Following [6, p. 27] we say that a property is satisfied *almost everywhere* in the irreducible variety V, or that it is satisfied at *almost all points* of V, if there exists a subvariety V_1 of V such that the property is verified at all the points of $V - V_1$.

THEOREM 4. *Let $f_t \in K[x_1,\dots,x_n]$ (K algebraically closed), where*

$$f_t = \sum_{1 \leqslant i_1 < \cdots < i_t \leqslant n} a_{i_1 \cdots i_t} x_{i_1} \cdots x_{i_t}, \qquad t = 1,\dots,n, \tag{21}$$

with $a_{i_1 \cdots i_t} \neq 0$, $t = 1,\dots,n$, $1 \leqslant i_1 < \cdots < i_t \leqslant n$. Let $(d_1,\dots,d_n)$ be an

n-tuple of elements of K. The system of n polynomial equations

$$f_t(x_1,\dots,x_n)+d_t=0, \qquad t=1,\dots,n, \tag{22}$$

has a solution. Moreover the number of solutions is finite and does not exceed $n!$, and if K is of characteristic 0 and there exists a point $(d_1,\dots,d_n)$ for which the system (22) has $n!$ solutions, then for almost all the points $(d_1,\dots,d_n)$ of K^n the number of solutions is exactly $n!$.

Proof. We have already seen, by Lemma 1, that there exist homogeneous polynomials

$$h_{ij}, \qquad i=1,\dots,n, \quad j=1,\dots,n,$$

and an integer q satisfying (9), (10), and (11). Let us define

$$g_i=f_i+d_i, \qquad i=1,\dots,n, \tag{23}$$

and

$$R_i=x_i^q+d_1h_{i1}+\cdots+d_nh_{in}, \qquad i=1,\dots,n. \tag{24}$$

Then R_i, $i=1,\dots,n$, is a polynomial of the form

$$R_i(x_1,\dots,x_n)=x_i^q+P_i(x_1,\dots,x_n)$$

with $\deg P_i<q$. From (9), (23), and (24) we get

$$R_i=g_1h_{i1}+g_2h_{i2}+\cdots+g_nh_{in}, \qquad i=1,\dots,n. \tag{25}$$

Therefore, denoting by H the matrix (h_{ij}), we obtain

$$\begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}=H\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}. \tag{26}$$

Denoting by $Z=(z_{ij})$ ($z_{ij}\in K[x_1,\dots,x_n]$) the adjoint of H, i.e.,

$$z_{ij}=(-1)^{i+j}\det H(j|i), \qquad i=1,\dots,n, \quad j=1,\dots,n,$$

and multiplying both sides of (26) by Z on the left, we obtain

$$\det Hg_t = \sum_{k=1}^{n} z_{tk}R_k \in (R_1,\dots,R_n), \qquad t=1,\dots,n.$$

If $g_1,\dots,g_n$ had no common zeros then by the Hilbert *Nullstellensatz* we would deduce that there exist polynomials

$$v_t \in K[x_1,\dots,x_n], \qquad t=1,\dots,n,$$

satisfying

$$1=\sum_{t=1}^{n} v_t g_t. \tag{27}$$

Multiplying both sides of (27) by $\det H$ we conclude that $\det H$ belongs to the ideal

$$(R_1,\dots,R_n),$$

which contradicts Lemma 3. So our system of equations has a solution. By the equalities (25) the solutions of the system (22) are also solutions of the system

$$R_i=0, \qquad i=1,\dots,n. \tag{28}$$

Since by Theorem 3 the number of solutions of (28) is finite, the number of solutions of (22) is also finite.

It remains to show that the number of solutions is no greater than $n!$, and if there exists a point $(d_1,\dots,d_n)$ for which the system (22) has $n!$ solutions, then for almost all the points $(d_1,\dots,d_n)$ of K^n the number of solutions is exactly $n!$. Following [3], let us consider the polynomials of $K[x_0,x_1,\dots,x_n]$

$$\hat{f}_t = f_t(x_1,\dots,x_n)+d_t x_0^t, \qquad t=1,\dots,n. \tag{29}$$

Then $\hat{f}_t$, $t=1,\dots,n$, are homogeneous polynomials of degree t. We consider now the system

$$\hat{f}_t=0, \qquad t=1,\dots,n. \tag{30}$$

Still following [3], and bearing in mind that the system $f_t = 0$, $t = 1, \ldots, n$, has only the trivial solution (as was observed in the proof of Lemma 1), we conclude that the system (30) has no nontrivial solutions with $x_0 = 0$. Thus all the nontrivial ray solutions are of the form (u, vz) where z is a solution of the system (22). Consequently, since we have already proved that the number of solutions of (22) is finite, we conclude that the number of ray solutions is finite. Therefore, by the Bézout theorem, it is less than or equal to $n!$. Again as a consequence of the Bézout theorem [6, p. 43], if there exists a point $(d_1, \ldots, d_n)$ for which the system (22) has $n!$ solutions, then (assuming now that K has characteristic 0) the number of solutions for almost all points $(d_1, \ldots, d_n)$ of K^n is exactly $n!$. ■

THEOREM 5. *Let A be an $n \times n$ matrix over the algebraically closed field K. Assume that all the principal minors of A are nonzero. For each n-tuple $(w_1, \ldots, w_n)$ of K^n there exists an n-square diagonal matrix D such that DA has eigenvalues $(w_1, \ldots, w_n)$. The number of matrices D is finite and does not exceed $n!$. Furthermore, if K is of characteristic 0 and there exists a point $(w_1, \ldots, w_n)$ for which there are $n!$ diagonal matrices D such that DA has eigenvalues $(w_1, \ldots, w_n)$, then for almost all n-tuples $(w_1, \ldots, w_n)$ the number of matrices D such that DA has eigenvalues $(w_1, \ldots, w_n)$ is exactly $n!$.*

Proof. This theorem is a consequence of the preceeding theorem and of the observations of Section 2. ■

After I had completed this paper I was informed by S. Friedland that my main result can be deduced in a different way by using the nonlinear alternative due to E. Noether and B. L. van der Waerden, as was done in Friedland's paper [3] for K = C. The result of Noether and van der Waerden was published in 1928 (Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. 1928:77–87). S. Friedland also published his paper [3] without knowing of the paper by Noether and van der Waerden [see Zbl. Math. 358:84–85 (1978)].

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